COMPUTATIONS IN HOCHSCHILD COHOMOLOGY OF GROUP ALGEBRAS

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Abstract. The Hochschild cohomology ring of a group algebra is an object that has received recent attention, but is difficult to compute, in even the simplest of cases. In this paper, we use the product formula due to Witherspoon and Siegel to extend some of their computations. In particular, we compute the Hochschild cohomology algebra of group algebras kG where $|G| \leq 15$ and we provide an alternative computation of the ring $HH^*(k(E \ltimes P))$ considered by Kessar and Linckelmann.

1. Background

We suppose throughout this paper that G is a finite group and k a field of characteristic p. The Hochschild cohomology ring $HH^*(kG)$ is defined as $\mathrm{Ext}^*_{kG\otimes kG^{\mathrm{op}}}(kG, kG)$ where kG has the obvious bimodule structure. Using the Eckmann-Shapiro Lemma, one sees that $HH^*(kG)$ and $H^*(G, {}_\psi kG)$ are isomorphic as algebras, where G acts on kG via conjugation. Furthermore, $H^*(G, {}_\psi kG)$ has a well-known additive decomposition. More precisely, let $\{g_i\}$ be representatives of the conjugacy classes of G, $H_i = C_G(g_i)$, and W_i the subspace of kG spanned by all conjugates of g_i, so that $W_i \simeq k_{H_i}\uparrow^G$. Since ${}_\psi kG = \bigoplus W_i$, the Eckmann-Shapiro Lemma yields a linear isomorphism

$$H^*(G, {}_\psi kG) = \bigoplus \mathrm{Ext}^*_{kG}(k, W_i) \simeq \bigoplus \mathrm{Ext}^*_{kH_i}(k,k) = \bigoplus H^*(H_i)$$

These results were known for some time, but it was only in [2] and [1] that the case of $HH^*(kG)$ for G an abelian group was fully completed. Using bimodule resolutions of kG as a G-module, one may compute $HH^*(kG)$ for several additional classes of groups, including the quaternions. However, an alternative method of computation is provided in [5]. There, it is determined how the additive decomposition behaves with respect to the cup product of $H^*(G, {}_\psi kG)$. In fact, they derive a product formula

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$$\gamma_i(\alpha) \smile \gamma_j(\beta) = \sum_{x\in D} \gamma_k(\mathrm{Tr}_W^{H_k}(\mathrm{Res}_W^{{}^yH_i}\, y^*\alpha \smile \mathrm{Res}_W^{{}^{yx}H_j}(yx)^*\beta)) \tag{1}$$

Here, γ_i is the composite $H^*(H_i) \to H^*(H_i, {}_\psi kG) \to H^*(G, {}_\psi kG)$, $D \subseteq G$ satisfies $G = \coprod_{x\in D} H_i x H_j$, for each $x \in D$ we have g_k conjugate with $g_i{}^x g_j$ with $g_k = {}^y(g_i{}^x g_j)$, and $W = {}^yH_i \cap {}^{yx}H_j$. They used this product formula, in particular to find finite presentations of the algebras $HH^*(kG)$ for several small examples and for dihedral 2-groups. The aim of this paper is to extend these computations in several directions.

The paper will proceed as follows. In section 2 we will introduce a modified version of $HH^*(kG)$ that is more convenient for computations but leaves unaltered the fundamental aspects of $HH^*(kG)$ and the product formula. Basically, we will modify the degree zero elements to ignore the redundant information they contain. In section 3 we provide the promised alternative proof to Kessar and Linckelmann's result from [3] using the product formula. In section 4 we proceed to compute finite presentations for $HH^*(kG)$ where $|G| \le 15$. Four of these cases were handled by the work performed in [5] and so this section is a continuation of the work started there.

2. Preliminary Results

There is often times much extraneous information given by $HH^0(kG)$. This occurs when H_i is a p'-group, since in this case $H^*(H_i) = k$ concentrated in degree zero. It is therefore useful to consider the Tate analogue $\widehat{HH}^*(kG)$ of Hochschild cohomology. For an algebra A, as per [3] we define

$$\widehat{HH}^*(A) = \bigoplus \overline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}(\Omega^n_{A\otimes A^{\mathrm{op}}}(A), A)$$

where $\overline{\mathrm{Hom}}_{A\otimes A^{\mathrm{op}}}$ denotes the homomorphism space in the stable module category ${}_{A\otimes A^{\mathrm{op}}}\overline{\mathrm{mod}}$, and Ω denotes the Heller operator. Recall that if $E_{A-A}(A)$ denotes the ring of all $A \otimes A^{\mathrm{op}}$-linear endomorphisms of A, then $Z(A) \simeq E_{A-A}(A)$ under the map $z \mapsto (a \mapsto az)$). There is an ideal $Z^{\mathrm{pr}}(A)$ of $Z(A)$ defined as the set of all endomorphisms of A that factor through an $A \otimes A^{\mathrm{op}}$-projective module. Then the stable center $\bar{Z}(A)$ is defined as $Z(A)/Z^{\mathrm{pr}}(A)$. It is easy to see that $\widehat{HH}^n(A) = HH^n(A)$ for $n > 0$ and that $\widehat{HH}^0(A) = \bar{Z}(A)$. Moreover, $\widehat{HH}^*(A)$ satisfies the analogue of Tate duality in the sense that $\widehat{HH}^{-n}(A) \simeq HH^{n-1}(A)^*$ for $n > 0$, where * denotes the dual space.

The next proposition justifies the assertion that $\widehat{HH}^*(kG)$ contains less extraneous information than does $HH^*(kG)$.

Proposition 2.1. Suppose G is a group with conjugacy class representatives $\{g_i\}$ and corresponding centralizers $H_i = C_G(g_i)$. If r is the number of indices i for which $p \mid |H_i|$, then $\dim \widehat{HH}^0(kG) = r$. Moreover, $\bar{Z}^{\mathrm{pr}}(kG) = \mathrm{Tr}_1^G(kG)$.

Proof. It suffices to show that $\dim \bar{Z}(kG) = r$. Recall that $kG \otimes kG^{\mathrm{op}} \simeq k[G \times G]$ with kG a $k[G \times G]$-module via the rule $(g,h)j = gjh^{-1}$. Notice that the map $\psi : k[G \times G] \twoheadrightarrow kG$ given by $\psi(g,h) = gh^{-1}$ is $k[G \times G]$-linear since $\psi((g_1,h_1)(g_2,h_2)) = g_1g_2h_2^{-1}h_1^{-1} = (g_1,h_1).\psi(g_2,h_2)$. So ψ is a projective cover, and hence an endomorphism of kG factors through a projective $k[G \times G]$-module precisely when it factors through ψ.

Now suppose $z \in Z(kG)$ and define an endomorphism f_z of kG via $f_z(x) = xz = zx$ for $x \in kG$. We aim to find a necessary and sufficient condition that f_z factor through ψ, in the sense that $f_z = \psi\sigma$ for some $\sigma : kG \to k[G \times G]$. Since $kG \simeq k_{\Delta G} \uparrow^{G \times G}$ as $k[G \times G]$-modules, we know that for every $k\Delta G$-linear map $k \to k[G \times G]$ there is a unique $k[G \times G]$-linear extension $kG \to k[G \times G]$. So σ is specified by $\sigma(1) = \sum_{(h,j)} \sigma_{(h,j)}(h,j)$ subject to $(g,g)\sigma(1) = \sigma(1)$ for all $g \in G$. In other words, $\sigma_{h,j} = \sigma_{gh,gj}$ for all $g,h,j \in G$. In particular, σ is determined by $\{\sigma_{g,1} : g \in G\}$ since $\sigma_{h,j} = \sigma_{j^{-1}h,1}$. From $f_z(1) = \psi(\sigma(1))$ we obtain $z = \sum \sigma_{h,j} hj^{-1}$. Let $\{g_i\}$ be conjugacy class representatives of G and for each g_i let $S_i = \{{}^g g_i : g \in G\}$. Define $\kappa_i = \sum_{g \in S_i} g$ and write $z = \sum z_i \kappa_i$ for scalars $z_i \in k$. Then

$$z_i = \sum_{j \in G} \sigma_{g_i j, j} = \sum_{j \in G} \sigma_{j^{-1} g_i j, 1} = |C_G(g_i)| \sum_{g \in S_i} \sigma_{g,1} \tag{2}$$

So a necessary and sufficient condition for f_z to factor through ψ is the existence of solutions to (2) for all i. If (2) is a consistent system and i is such that $p \mid |H_i|$, then $z_i = 0$, so that z is a combination of all κ_i with $p \nmid |H_i|$. Conversely, if z is of this form then we may find a solution to (2) by setting, for example, $\sigma_{g,1} = 0$ if $p \mid |H_i|$ and $\sigma_{g,1} = z_i/|H_i|$ if $p \nmid |H_i|$. Therefore

$$\dim Z^{\mathrm{pr}}(kG) = |\{i : p \nmid |H_i|\}| = \dim Z(kG) - r$$

and so $\dim \bar{Z}(kG) = \dim Z(kG) - \dim Z^{\mathrm{pr}}(kG) = r$ as claimed. The equality $\bar{Z}^{\mathrm{pr}}(kG) = \mathrm{Tr}_1^G(kG)$ follows from the fact that $\mathrm{Tr}_1^G(g_i) = 0$ if $p \mid |C_G(g_i)|$ while $\mathrm{Tr}_1^G(\frac{1}{|C_G(g_i)|} g_i) = \kappa_i$ if $p \nmid |C_G(g_i)|$. □

Notice that $\widehat{HH}^*(kG) = \bigoplus \widehat{H}^*(H_i)$ as graded vector spaces where $\hat{H}^*(H_i)$ is the Tate cohomology of H_i, and that (1) still makes sense since restriction and trace are defined for Tate cohomology of groups. It is not clear whether the product formula (1) remains valid in this case, but even if it did, there is good reason to avoid computing products in negative degrees since these products may be complicated. Our original goal in introducing $\widehat{HH}^*(kG)$ was simply in discarding the extraneous information present in degree zero, and so we are justified in merely considering the subalgebra $HH'(kG) := \bigoplus_{n\geq 0} \widehat{HH}^n(kG)$. Note that $Z^{\mathrm{pr}}(kG)$ is an ideal in $Z(kG)$ and $HH^*(kG)$ as well. After all, if $\alpha \in H^*(H_i)$ and $\beta \in H^*(H_j)$ with $\deg\beta > 0$ and $p \nmid |H_i|$, then $H^{\deg\beta}(W) = 0$ in the notation of (1), so that $\gamma_i(\alpha) \smile \gamma_j(\beta) = 0$. So $Z^{\mathrm{pr}}(kG)$ is an ideal in $HH^*(kG)$, $HH'(kG) = HH^*(kG)/Z^{\mathrm{pr}}(kG)$, and hence formula (1) remains valid for $HH'(kG)$.

Remark. One of the original motivations for our computations was to investigate a conjecture formulated in [5] regarding a relationship between $HH^*(kG)$ and $H^*(G)$. More precisely, let e_0 be the principal block idempotent of kG with $b_0 = e_0kG$, and recall that $HH^*(b_0)$ is an ideal direct summand of $HH^*(kG)$ with $HH^*(b_0) = e_0HH^*(kG)$. Then the composite map $H^*(G) \to HH^*(kG) \twoheadrightarrow HH^*(b_0)$ is a monomorphism with retraction $HH^*(b_0) \hookrightarrow HH^*(kG) \to H^*(G)$, where the map $HH^*(kG) = H^*(G, kG) \to H^*(G)$ is induced by $\varepsilon : kG \to k$. It was conjectured that the composite is an isomorphism module radicals; a result that was later established by Linckelmann in [4]. In this regard, our modification $HH'(kG)$ is harmless since the monomorphism $H^*(G) \hookrightarrow HH'(kG)$ remains an isomorphism modulo radicals. After all, the only thing to check is that $Z(b_0)/Z^{\mathrm{pr}}(b)$ and k are isomorphic modulo radicals, which is true since $J(Z(b_0))$ has codimension 1 in $Z(b_0)$.

The next two results will prove indispensable, where the first is a standard application of the LHS.

Lemma 2.2. Suppose G is a group with subgroup $N \triangleleft G$ and k is a field of characteristic p. If $p \nmid |N|$ then $H^*(G) \simeq H^*(G/N)$. If $p \nmid |G : N|$ then $\mathrm{Res} : H^*(G) \simeq H^*(N)^G$ is an isomorphism with inverse $\frac{1}{|G:N|}\mathrm{Tr}_N^G$. Moreover, if we identify $H^*(G)$ with $H^*(N)^G$ and let $\alpha \in H^*(N)$, then $\mathrm{Tr}_N^G(\alpha) = \sum_{g\in G/N} g^*\alpha$.

Lemma 2.3. Suppose G has a Sylow p-subgroup P and a normal p-complement Q. If k is a field with characteristic p, then $\mathrm{Res} : H^*(G) \to H^*(P)$ is an isomorphism.

Proof. If $e = \frac{1}{|Q|}\sum_{q\in Q} q$ so that ekG is the principal block of kG. Then $H^*(kG) \simeq \mathrm{Ext}^*_{ekG}(k,k) \simeq H^*(P)$ since $ekG \simeq kP$. Since Res : $H^*(G) \to H^*(P)$ is injective, we see that Res is an isomorphism. □

3. Linckelmann's Result

We now use the product formula to establish a modified form of Proposition 5.2 from [3].

Theorem 3.1. Suppose P is a nontrivial abelian p-group and E is a p'-subgroup of $\mathrm{Aut}(P)$ that acts semiregularly on $P\setminus\{1\}$. Then E acts on the algebra $kP \otimes H^*(P)$ diagonally via automorphisms, and there is an isomorphism of k-algebras

$$HH'(k(E \ltimes P)) \simeq (kP \otimes H^*(P))^E$$

Proof. Define $G = E \ltimes P$ and note that if $H \leq G$ then $H \cap P$ is a Sylow p-subgroup of H, and hence Res : $H^*(H) \to H^*(P)^{E_H}$ is an isomorphism by Lemma 2.2, where E_H is a p-complement in H. Of course $E_G = E$ and so $H^*(G) \simeq H^*(P)^E$. For $g \in G$ write $g = pe$ with $p \in P$ and $e \in E$. If $p_1 \in P$ then $p_1 g = pp_1 e$ since P is abelian, and $gp_1 = p^e p_1 e$, so that $p_1 g = gp_1$ precisely when $p_1 = {}^e p_1$. That is, $C_G(g) \cap P = C_G(e) \cap P$. Since E acts semiregularly on $P \setminus \{1\}$, $C_G(e) \cap P = 1$ for $e \neq 1$ and of course $C_G(1) \cap P = P$. Therefore, $C_G(g) \cap P = 1$ for $g \notin P$ and $C_G(g) = P$ for $1 \neq g \in P$. So if G has conjugacy class representatives $\{g_i\}$, then $H^*(g_i) = k$ in degree zero whenever $g_i \notin P$, and $H^*(g_i) = H^*(P)$ whenever $g_i \in P$. Also, the set of all g_i that belong to P comprise the representatives of the orbits of E acting on P. The additive decomposition for $HH'(kG)$ now becomes

$$HH'(kG) = \bigoplus_{p \nmid |H_i|} H^*(C_G(g_i)) = H^*(P)^E \oplus \bigoplus_{1 \neq g_i \in P} H^*(P)$$

Define $\Psi : HH'(kG) \to kP \otimes H^*(P)$ by

$$\Psi(\gamma_i(\alpha)) = \begin{cases} 1 \otimes \alpha & \text{if } g_i = 1 \\ \sum_{e\in E} {}^e g_i \otimes e^*\alpha & \text{if } 1 \neq g_i \in P \end{cases}$$

Clearly $\Psi(\gamma_1(\alpha)) \in (kP \otimes H^*(P))^E$ for $\alpha \in H^*(P)^E$, and also $\Psi(\gamma_i(\alpha)) \in (kP \otimes H^*(P))^E$ for $\alpha \in H^*(P)$ since

$${}^f\Psi(\gamma_i(\alpha)) = \sum_{e\in E} {}^{fe} g_i \otimes (fe)^*\alpha = \sum_{e\in E} {}^e g_i \otimes e^*\alpha = \Psi(\gamma_i(\alpha))$$

whenever $f \in E$. It is clear that Ψ is injective. Notice that $kP \otimes H^*(kP)$ is graded by taking kP concentrated in degree zero. Since E preserves the grading of $kP \otimes H^*(kP)$, an element $\xi \in kP \otimes H^*(kP)$ is fixed under E if and only if each of its homogeneous components is fixed under E. Also, any homogeneous element ξ of $kP \otimes H^*(kP)$ of degree n may be expressed as

$$\xi = 1 \otimes \alpha + \sum_{1 \neq g_i \in P} \sum_{e \in E} {}^e g_i \otimes \alpha_{i,e}$$

for some $\alpha, \alpha_{i,e} \in H^*(kP)$. It is clear that ξ is fixed under E precisely when $\alpha \in H^*(P)^E$ and $f^*\alpha_{i,e} = \alpha_{i,fe}$ for all $e, f \in E$. In other words, $\alpha_{i,e} = e^*\alpha_{i,1}$ for all $e \in E$. Therefore, Ψ is a surjective map, and it remains to verify that Ψ is multiplicative. It suffices to show that $\Psi(\gamma_i(\alpha)\gamma_j(\beta)) = \Psi(\gamma_i(\alpha))\Psi(\gamma_j(\beta))$. This is clear if i or j equals 1 since $\gamma_1(\alpha)\gamma_i(\beta) = \gamma_i(\mathrm{Res}(\alpha)\beta)$. So assume $1 \neq g_i, g_j \in P$ and $\alpha, \beta \in H^*(P)$. Note that $D = E$ in the notation of (1) and so for every $e \in E$ there is a summand $\gamma_k(\cdot)$ in the product $\gamma_i(\alpha)\gamma_j(\beta)$, and hence a corresponding summand $\Psi(\gamma_k(\cdot))$ in $\Psi(\gamma_i(\alpha)\gamma_j(\beta))$. More precisely, for $e \in E$ there is $e' \in E$ and $g_k \in P$ with $g_k = {}^{e'}(g_i {}^e g_j)$. If $g_k \neq 1$ then the summand in $\gamma_i(\alpha)\gamma_j(\beta)$ equals $\gamma_k((e')^*\alpha \smile (e'e)^*\beta)$ and so the corresponding summand in $\Psi(\gamma_i(\alpha)\gamma_j(\beta))$ equals

$$\begin{aligned}\Psi(\gamma_k((e')^*\alpha \smile (e'e)^*\beta)) &= \sum_{f \in E} {}^f({}^{e'}(g_i {}^e g_j)) \otimes f^*((e')^*\alpha \smile (e'e)^*\beta) \\ &= \sum_{f \in E} {}^f(g_i {}^e g_j) \otimes (f^*\alpha \smile (fe)^*\beta)\end{aligned} \tag{3}$$

since fe' ranges across E as f ranges across E. On the other hand, if $g_k = 1$ then the summand in $\gamma_i(\alpha)\gamma_j(\beta)$ equals $\gamma_1(\mathrm{Tr}_P^G(\alpha \smile e^*\beta)) = \sum_{f \in E} \gamma_1(f^*(\alpha \smile e^*\beta))$ by Lemma 2.2. So the corresponding summand in $\Psi(\gamma_i(\alpha)\gamma_j(\beta))$ equals

$$\sum_{f \in E} {}^f(g_i {}^e g_j) \otimes (f^*\alpha \smile (fe)^*\beta) \tag{4}$$

since ${}^f(g_i {}^e g_j) = {}^f 1 = 1$ for $f \in E$. As (f, e) ranges across $E \times E$ so does $(e_1, e_2) := (f, fe)$. So summing (3) and (4) across $e \in E$ yields

$$\begin{aligned}\Psi(\gamma_i(\alpha)\gamma_j(\beta)) &= \sum_{f\in E}\sum_{e\in E} {}^{f}(g_i{}^{e}g_j)\otimes(f^*\alpha\smile(fe)^*\beta)\\ &= \sum_{e_1,e_2\in E} {}^{e_1}g_i{}^{e_2}g_j\otimes(e_1^*\alpha\smile e_2^*\beta)=\Psi(\gamma_i(\alpha))\Psi(\gamma_j(\beta))\end{aligned}$$

as desired. So the proof is complete. □

4. Groups of Order less than 16

We may now 'complete' the computation of $HH'(kG)$ for $|G|\leq 15$ begun in [5]. We simply need to compute HH' for $\mathbb{F}_2\,D_{10}$, $\mathbb{F}_3\,A_4$, $\mathbb{F}_2\,D_{12}$, $\mathbb{F}_3\,D_{12}$, $\mathbb{F}_2\,T$, $\mathbb{F}_3\,T$, and $\mathbb{F}_2\,D_{14}$, where T denotes the non-abelian group of order 12 not isomorphic with D_{12} or A_4. Recall that $HH'(kG)=\bigoplus_{p||H_i|}H^*(H_i)$ where $H_i=C_G(g_i)$. To describe a finite presentation for $HH'(kG)$ we follow [5] by listing homogeneous elements $X_1,\ldots,X_m$ that generate $H^*(G)\simeq\gamma_1(H^*(G))$ as an algebra subject to the homogenous relations $r_1,\ldots,r_n$; homogenous elements $Y_1,\ldots,Y_{m'}\in\bigoplus_{i\geq 2}H^*(H_i)$ that generate $\bigoplus_{i\geq 2}H^*(H_i)$ as a $\gamma_1(H^*(G))$-module subject to relations $r_1,\ldots,r_{n'}$; and relations of the form $Y_iY_j=X_{ij}+\sum X_{ij}^kY_k$ for all i and j where $X_{ij},X_{ij}^k\in\gamma_1(H^*(G))$. We refer to these as relations of Type I, Type II, and Type III respectively. Together with the (implicitly understood) graded-commutative relations, we obtain an abstract presentation of the algebra $HH'(kG)$. In performing these computations it is convenient to identify $HH'(kG)$ in degree zero with $\bar{Z}(kG)$.

Proposition 4.1. For n odd, $HH'(\mathbb{F}_2\,D_{2n})$ has generators C_1,C_2, and V of degrees 0,0, and 1, respectively, subject to the relation $(C_2)^2=C_1$.

Proof. Write $n=2l+1$ and $D_{2n}=\langle a,b|a^n=b^2=1,bab^{-1}=a^{-1}\rangle$. Note that D_{2n} has conjugacy class representatives $\{g_i\}=\{1,a,\ldots,a^l,b\}$ with corresponding centralizers $\{H_i\}=\{D_{2n},\langle a\rangle,\ldots,\langle a\rangle,\langle b\rangle\}$. Lemma 2.3 implies that Res : $H^*(D_{2n})\to H^*(\langle b\rangle)$ is an isomorphism, and so $H^*(D_{2n})$ is generated by an element v of degree 1. Therefore, $HH'(\mathbb{F}_2\,D_{2n})$ has generators $C_1=\gamma_1(1),C_2=\gamma_2(1)$, and $V=\gamma_1(v)$ of degrees 0,0,and 1, respectively. The only relations are of Type III in degree zero. Identifying $HH'^0(kG)$ with $\bar{Z}(kG)$ and defining $\tau=\sum_{i=0}^{n-1}a^i$, this relation takes the form

$$(C_2)^2=(\tau b+Z^{\mathrm{pr}}(kG))^2=\tau^2+Z^{\mathrm{pr}}(kG)=\tau+Z^{\mathrm{pr}}(kG)=1+Z^{\mathrm{pr}}(kG)=C_1$$

So we have a complete presentation of $HH'(kG)$. □

Proposition 4.2. $HH'(\mathbb{F}_3 A_4)$ has generators C_1, C_2, C_3, V, and U of degrees 0,0,0,1, and 2, respectively, subject to the relations $(C_2)^2 = C_3, (C_3)^2 = C_2$, and $C_2C_3 = C_3C_2 = C_1$.

Proof. Let $a = (12)(34), b = (13)(24)$, and $c = (123)$. So A_4 has conjugacy class representatives $\{g_i\} = \{1, c, c^2, a\}$ with corresponding centralizers $\{H_i\} = \{A_4, \langle c\rangle, \langle c\rangle, P\}$ where $P = \langle a, b\rangle$ is a normal Sylow 2-subgroup of A_4. Lemma 2.3 implies that $\mathrm{Res} : H^*(A_4) \to H^*(\langle c\rangle)$ is an isomorphism. So $H^*(A_4) = \mathbb{F}_3[u, v]$ where $\deg u = 2$ and $\deg v = 1$. Of course, $H^*(\langle c\rangle)$ is free as an $H^*(A_4)$-module with basis 1. Hence, $HH'(\mathbb{F}_3 A_4)$ has generators $C_1 = \gamma_1(1), C_2 = \gamma_2(1), C_3 = \gamma_3(1), V = \gamma_1(v)$, and $U = \gamma_1(u)$ of degrees 0,0,0,1, and 2, respectively. There are no nontrivial Type I or Type II relations. The Type III relations only occur in degree zero, in which case they are $(C_2)^2 = C_3, (C_3)^2 = C_2$, and $C_2C_3 = C_1$, as is straightforward to check. $\square$

Theorem 4.3. $HH'(\mathbb{F}_2 D_{12})$ has generators C_1,C_2,C_3,C_4,C_5,C_6,V_1, and V_2 of degrees 0,0,0,0,0,0,1, and 1, respectively, subject to the relations $V_2C_3 = V_2C_4 = 0$ and those given by the table below:

	C_2	C_3	C_4	C_5	C_6
C_2	C_1	C_4	C_3	C_6	C_5
C_3	C_4	C_4	C_3	0	0
C_4	C_3	C_3	C_4	0	0
C_5	C_6	0	0	$C_1 + C_4$	$C_2 + C_3$
C_6	C_5	0	0	$C_2 + C_3$	$C_1 + C_4$

Moreover, $HH'(\mathbb{F}_3 D_{12})$ has generators $C_1, C_2, C_3, C_4, X_3, X_4, Y_3, Y_4, V$, and U of degrees 0,0,0,0,1,1,2,2,3, and 4, respectively, subject to the Type II relations $VX_3 = VX_4 = 0$, $UX_3 = VY_3$, and $UX_4 = VY_4$; Type III relations in degree zero given as $(C_2)^2 = C_1$, $(C_3)^2 = C_4 - C_1$, $(C_4)^2 = C_4 - C_1$, $C_2C_3 = C_4$, $C_2C_4 = C_3$, and $C_3C_4 = C_3 - C_2$; and the Type III relations in positive degree given in the following table.

	X_3	Y_3	X_4	Y_4
C_2	$2X_4$	$2Y_4$	$2X_3$	$2Y_3$
C_3	X_4	Y_4	X_3	Y_3
C_4	$2X_3$	$2Y_3$	$2X_4$	$2Y_4$
X_3	0	$VC_4 + V$	0	$2VC_2 + 2VC_3$
Y_3	$VC_4 + V$	$UC_4 + U$	$2VC_2 + 2VC_3$	$2UC_2 + 2UC_3$
X_4	0	$2VC_2 + 2VC_3$	0	$VC_4 + V$
Y_4	$2VC_2 + 2VC_3$	$2UC_2 + 2UC_3$	$VC_4 + V$	$UC_4 + U$

Proof. Let $G = D_{12} = \langle a, b | a^6 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ and define $N = \langle a\rangle$ and $Z = Z(G) = \langle a^3\rangle$. So G has conjugacy class representatives $\{g_i\} = \{1, a^3, a, a^2, b, ab\}$ with corresponding centralizers $\{G, G, N, N, \langle a^3, b\rangle, \langle a^3, ab\rangle\}$ and orders $\{12, 12, 6, 6, 4, 4\}$.

($p = 2$) Since $\langle a^2\rangle$ is a normal 2-complement, Lemma 2.3 implies that $\mathrm{Res} : H^*(G) \to H^*(P)$ is an isomorphism whenever $P \in \mathrm{Syl}_2(G)$, and in particular $H^*(P)$ is free as an $H^*(G)$-module with basis 1. Fix $P = \langle a^3, b\rangle$ and choose generators v_1 and v_2 of $H^*(G)$ with degrees 1 and 1, so that $\mathrm{Res}^G_P(v_1)$ and $\mathrm{Res}^G_P(v_2)$ are 'dual' to a^3 and b, in the sense that $\mathrm{Res}^G_Z(v_1) \neq 0$ and $\mathrm{Res}^G_Z(v_2) = 0$. From $\mathrm{Res}^N_Z \mathrm{Res}^G_N = \mathrm{Res}^P_Z \mathrm{Res}^G_P$ and $\mathrm{Tr}^N_Z \mathrm{Res}^N_Z = |N : Z| \,\mathrm{Id} = \mathrm{Id}$ we obtain $\mathrm{Res}^G_N = \mathrm{Tr}^N_Z \mathrm{Res}^P_Z \mathrm{Res}^G_P$. Since Tr^N_Z is an isomorphism by Lemma 2.3, $\mathrm{Res}^G_N(v_1)$ generates $H^*(N)$ as an algebra, and $\mathrm{Res}^G_N(v_2) = 0$. In particular, $H^*(N)$ is generated by 1 as an $H^*(G)$-module. Therefore, we have generators of $HH'(\mathbb{F}_2\, D_{12})$ given by $C_i = \gamma_i(1)$ for $1 \le i \le 6$, and $V_j = \gamma_1(v_j)$ for $j = 1, 2$. There are no nontrivial Type I relations, Type III relations occur only in degree zero, and we have the relations $V_2C_3 = V_2C_4 = 0$ of Type II. If we let $H^*(G)[n]$ denote the free $H^*(G)$-module with grading $H^i(G)[n] = H^{i-n}(G)$, then we have a graded complex of $H^*(G)$-modules given by

$$H^*(G)[1] \xrightarrow{\delta_1} H^*(G) \xrightarrow{\delta_0} H^*(N) \longrightarrow 0$$

where $\delta_0(\alpha) = \alpha.1$ and $\delta_1(\alpha) = \alpha v_2$ for $\alpha \in H^*(G)$. Note that δ_0 is surjective and δ_1 is injective since $H^*(G)$ is an integral domain. Since $\dim H^i(G)[1] - \dim H^i(G) + \dim H^i(N) = 0$ for all $i \geq 0$, we see that this complex is exact, and hence we have a free presentation of $H^*(N)$ as an $H^*(G)$-module. In particular, we have accounted for all Type II relations.

($p = 3$) Lemma 2.2 implies that $\mathrm{Res} : H^*(G) \to H^*(N)^G$ is an isomorphism. So there are generators x and y of $H^*(N)$ with degrees 1 and 2, and generators v and u of $H^*(G)$ with degrees 3 and 4, such that restriction $H^*(G) \to H^*(N)$ sends v to xy and u to y^2. In particular, $H^*(N)$ is generated as an $H^*(G)$-module by $1, x,$ and y, with relations $v.x = 0$ and $u.x = v.y$. So letting $B[n]_* = H^*(G)[n]$ denote the free graded $H^*(G)$-module of rank 1, we have a graded complex of $H^*(G)$-modules given by

$$B[4] \oplus B[5] \xrightarrow{\delta_1} B \oplus B[1] \oplus B[2] \xrightarrow{\delta_0} H^*(N) \longrightarrow 0$$

where

$$\delta_0(\alpha_1, \alpha_2, \alpha_3) = \alpha_1.1 + \alpha_2.x + \alpha_3.y$$
$$\delta_1(\alpha_1, \alpha_2) = \alpha_1(0, v, 0) + \alpha_2(0, u, -v)$$

Clearly δ_0 is surjective. Suppose $(\alpha_1, \alpha_2) \in (B[4] \oplus B[5])_i$ and $\delta_1(\alpha_1, \alpha_2) = 0$, so that $\alpha_2 v = 0$ and $\alpha_2 u + \alpha_1 v = 0$. Homogeneous elements of even order are non-zero divisors in $H^*(G)$, whereas odd degree elements are only annihilated by other odd degree elements. So $\alpha_2 = \lambda v u^j$ for some $\lambda \in k$ and $j \geq 0$. Also, from $0 = v(\alpha_1 + \lambda u^{j+1})$ we obtain $\alpha_1 = -\lambda u^{j+1}$ and so $i = 4j + 8$. Therefore, $\mathrm{Ker}(\delta_1) = \bigoplus_{j=0}^{\infty} \mathrm{Ker}(\delta_1|(B[4] \oplus B[5])_{4j+8})$ where $\mathrm{Ker}(\delta_1|(B[4] \oplus B[5])_{4j+8})$ is spanned by $(-u^{j+1}, vu^j)$. Using the fact that $\dim B[n]_t$ has Poincaré series $q_n(t)$ given as

$$q_n(t) = t^n(1 + t^3 + t^4 + t^7 + t^8 + \cdots)$$

it is easy to see that the sequence is exact, and hence we have found a free presentation of $H^*(N)$ as an $H^*(G)$-module. Thus, $HH'(\mathbb{F}_3 D_{12})$ has generators given by $C_i = \gamma_i(1)$ for $1 \leq i \leq 4$, $V = \gamma_1(v)$, $U = \gamma_1(u)$, and $X_j = \gamma_j(x)$ and $Y_j = \gamma_j(y)$ for $j = 3, 4$. There are no nontrivial Type I relations. The Type II relations are $VX_3 = VX_4 = 0$, $UX_3 = VY_3$, and $UX_4 = VY_4$. Type III relations in degree zero are obtained in the usual fashion, and Type III relations in positive degrees are obtained from the rules:

$$\gamma_3(\alpha) \smile \gamma_3(\beta) = \gamma_4(\alpha \smile \beta) + \gamma_1(\mathrm{Tr}_N^G(\alpha \smile b^*\beta))$$
$$\gamma_4(\alpha) \smile \gamma_4(\beta) = \gamma_4(b^*\alpha \smile b^*\beta) + \gamma_1(\mathrm{Tr}_N^G(\alpha \smile b^*\beta))$$
$$\gamma_3(\alpha) \smile \gamma_4(\beta) = \gamma_2(\mathrm{Tr}_N^G(\alpha \smile \beta)) + \gamma_3(b^*\alpha \smile \beta))$$

Note that $b^*x = x$ and $b^*y = y$; this is how we computed $H^*(G) \simeq H^*(N)^G$. So we have a complete presentation of $HH'(\mathbb{F}_3 D_{12})$. $\square$

Theorem 4.4. Suppose $T = \mathbb{Z}_4 \ltimes \mathbb{Z}_3$. Then $HH'(\mathbb{F}_2 T)$ has generators C_1, C_2, C_3, C_4, C_5, C_6, U, V, X_1 and X_2 of degrees 0,0,0,0,0,0,1,2,1, and 1, respectively, subject to the Type I relation $V^2 = 0$, the Type II relations $VC_3 = VC_4 = VX_1 = VX_2 = 0$, the Type III relations in degree zero given as follows:

	C_2	C_3	C_4	C_5	C_6
C_2	C_1	C_4	C_3	C_6	C_5
C_3	C_4	C_3	C_4	0	0
C_4	C_3	C_4	C_3	0	0
C_5	C_6	0	0	C_2+C_4	C_1+C_3
C_6	C_5	0	0	C_1+C_3	C_2+C_4

and the Type III relations in positive degrees given by following table:

	C_2	C_3	C_4	C_5	C_6	X_3	X_4
X_3	X_4	X_3+V	X_4	VC_5	VC_6	UC_3	UC_4
X_4	X_3	X_3+V	X_4	VC_6	VC_5	UC_4	UC_3

Moreover, $HH'(\mathbb{F}_3 T)$ has generators $C_1, C_2, C_3, C_4, U, V, X_1, X_2, Y_1$ and Y_2 of degrees 0,0,0,0,3,4,1,2,1, and 2, respectively, subject to the Type III relations in degree zero given as follows:

	C_2	C_3	C_4
C_2	C_1	C_4	C_3
C_3	C_4	$2C_1+C_3$	$2C_2+C_4$
C_4	C_3	$2C_2+C_4$	$2C_1+C_3$

and the Type III relations in positive degrees given as

	C_2	C_3	C_4	X_3	Y_3	X_4	Y_4
X_3	X_4	$2X_3$	$2X_4$	0	VC_3+V	0	VC_4+VC_2
Y_3	Y_4	$2Y_3$	$2Y_4$	VC_3+V	UC_3+U	VC_4+VC_2	UC_4+UC_2
X_4	X_3	$2X_4$	$2X_3$	0	VC_4+VC_2	0	VC_3+V
Y_4	Y_3	$2Y_4$	$2Y_3$	VC_4+VC_2	UC_4+UC_2	VC_3+V	UC_3+U

Proof. Write $T = \langle a, b | a^4 = b^3 = 1, aba^{-1} = b^{-1}\rangle$ and $N = \langle a^2, b\rangle$ so that T has conjugacy class representatives $\{g_i\} = \{1, a^2, b, a^2b, a, a^3b\}$ with corresponding centralizers $\{H_i\} = \{T, T, N, N, \langle a\rangle, \langle a^3b\rangle\}$ with orders $\{|H_i|\} = \{12, 12, 6, 6, 4, 4\}$.

($p = 2$) Lemma 2.3 implies that $\mathrm{Res} : H^*(T) \to H^*(P)$ is an isomorphism whenever $P \in \mathrm{Syl}_2(T)$. So $H^*(T)$ is generated by elements v and u of degrees 1 and 2, respectively, subject to the relation $v^2 = 0$. Also, $H^*(N) \simeq H^*(\langle a^2 \rangle)$ is generated by an element x with degree 1. As usual, $H^*(P)$ is the free $H^*(T)$-module with basis 1. Since $\mathrm{Res} : H^*(\langle a \rangle) \to H^*(\langle a^2 \rangle)$ is zero in odd degrees and nonzero in even degrees, we see that $\mathrm{Res} : H^*(T) \to H^*(N)$ maps v to 0 and u to x^2. Hence, $H^*(N)$ is generated as an $H^*(T)$-module by 1 and x. So letting $B[n]_* = H^*(T)[n]$ denote the free graded $H^*(T)$-module of rank 1, we have a graded complex of $H^*(T)$-modules given by

$$B[1] \oplus B[2] \xrightarrow{\delta_1} B[0] \oplus B[1] \xrightarrow{\delta_0} H^*(N) \longrightarrow 0$$

where

$$\begin{aligned} \delta_0(\alpha_1, \alpha_2) &= \alpha_1.1 + \alpha_2.x \\ \delta_1(\alpha_1, \alpha_2) &= \alpha_1(v, 0) + \alpha_2(0, v) \end{aligned}$$

Clearly, δ_0 is surjective and $\delta_1(\alpha_1, \alpha_2) = 0$ precisely when α_1 and α_2 have odd degree. So if $q(t) = \sum_{i=0}^{\infty} t^i$ then $B[0] \oplus B[1]$ has corresponding Poincaré polynomial $q(t) + tq(t)$, $\mathrm{Im}(\delta_1)$ has polynomial $tq(t)$, and $H^*(N)$ has polynomial $q(t)$, so that the above sequence is a free presentation of $H^*(N)$ as an $H^*(T)$-module. Therefore, we can choose generators of $HH'(kG)$ by $C_i = \gamma_i(1)$ for $1 \leq i \leq 6$, $V = \gamma_1(v)$, $U = \gamma_1(u)$, and $X_j = \gamma_j(x)$ for $j = 3, 4$. The only Type I relation is $V^2 = 0$, the Type II relations are $VC_3 = VC_4 = VX_3 = VX_4 = 0$, and the Type III relations in degree zero are handled in the usual way. The Type III relations in positive degree are obtained by using (1). For instance, we have the following:

$$\begin{aligned} \gamma_3(\alpha) \smile \gamma_2(\beta) &= \gamma_4(\alpha \smile \mathrm{Res}^T_H(\beta)) \\ \gamma_3(\alpha) \smile \gamma_3(\beta) &= \gamma_3(a^*\alpha \smile a^*\beta) + \gamma_1(\mathrm{Tr}^T_N(\alpha \smile a^*\beta)) \\ \gamma_3(\alpha) \smile \gamma_4(\beta) &= \gamma_4(a^*\alpha \smile a^*\beta) + \gamma_2(\mathrm{Tr}^T_N(\alpha \smile a^*\beta)) \\ \gamma_4(\alpha) \smile \gamma_4(\beta) &= \gamma_3(a^*\alpha \smile a^*\beta) + \gamma_1(\mathrm{Tr}^T_N(\alpha \smile a^*\beta)) \end{aligned}$$

From $\mathrm{Tr}^T_N = \mathrm{Tr}^T_P \mathrm{Tr}^P_Z \mathrm{Res}^N_Z$ we obtain $\mathrm{Tr}^T_N(1) = \mathrm{Tr}^T_N(y) = 0$ and $\mathrm{Tr}^T_N(x) = v$. Also, $a^*\alpha = \alpha$ for $\alpha \in H^*(Z)$ since $Z \leq ZT$. The computations are now laborious but straightforward.

($p = 3$) Lemma 2.2 implies that $\mathrm{Res} : H^*(T) \to H^*(N)^T$ is an isomorphism. So $H^*(T)$ is generated by elements v and u of degrees 3 and 4, respectively. We may choose generators x and y of $H^*(N)$ with degrees 1 and 2, respectively, for which $\mathrm{Res}(v) = xy$ and $\mathrm{Res}(u) = y^2$.

Then $H^*(N)$ is generated by 1, x, and y as an $H^*(T)$-module, with relations $v.x = 0$ and $u.x = v.y$. In fact, this provides a free presentation of $H^*(N)$ as an $H^*(T)$-module. We obtain generators of $HH'(kG)$ by defining $C_i = \gamma_i(1)$ for $1 \leq i \leq 4$, $V = \gamma_1(v)$, $U = \gamma_1(u)$, $X_j = \gamma_j(x)$, and $Y_j = \gamma_j(y)$ for $j = 3, 4$. There are no nontrivial Type I relations, and the Type II relations are given by $VX_3 = VX_4 = 0$, $UX_3 = VY_3$, and $UX_4 = VY_4$. Type III relations are obtained by the same means as for $p = 2$. In fact, (1) remains valid regardless of the characteristic of the field. For these computations it useful to have the following table at hand:

ξ	1	x	y	xy	y^2
$\mathrm{Tr}_N^T(\xi)$	2	0	0	$2v$	$2u$

The proof is now complete. □

We include one last computation which is interesting in that it requires few detailed multiplications.

Proposition 4.5. *$HH'(\mathbb{F}_3\,\mathrm{SL}_2(3))$ has generators $C_1, C_2, C_3, C_4, C_5, C_6, V$, and U of degrees 0,0,0,0,0,0,1, and 2, respectively, subject to the relations given by the following table:*

	C_2	C_3	C_4	C_5	C_6
C_2	C_1	C_6	C_5	C_4	C_3
C_3	C_6	C_4	C_1	C_2	C_5
C_4	C_5	C_1	C_3	C_6	C_2
C_5	C_4	C_2	C_6	C_3	C_1
C_6	C_3	C_5	C_2	C_1	C_4

Proof. Recall that $|\mathrm{SL}_2(3)| = 24$ and the Sylow 2-subgroup of $\mathrm{SL}_2(3)$ is normal. Thus, Lemma 2.3 implies that $\mathrm{Res} : H^*(\mathrm{SL}_2(3)) \to H^*(P)$ is an isomorphism for all $P \in \mathrm{Syl}_3(\mathrm{SL}_2(3))$. In particular, $H^*(\mathrm{SL}_2(3))$ has generators v and u of degrees 1 and 2, respectively. Direct computation shows that $\mathrm{SL}_2(3)$ has conjugacy class representatives $\{g_i\}$ given as

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \begin{pmatrix}2&0\\0&2\end{pmatrix}\ \begin{pmatrix}0&1\\2&2\end{pmatrix}\ \begin{pmatrix}2&2\\1&0\end{pmatrix}\ \begin{pmatrix}0&1\\2&1\end{pmatrix}\ \begin{pmatrix}1&2\\1&0\end{pmatrix}\ \begin{pmatrix}0&1\\2&0\end{pmatrix}$$

with centralizers $H_1 = H_2 = \mathrm{SL}_2(3)$ and $H_i = \langle g_i, 2I\rangle$ for $3 \leq i \leq 7$. So H_i is abelian for $3 \leq i \leq 7$, $|H_i| = 6$ for $3 \leq i \leq 6$, and $|H_7| = 4$. In particular, if P_i denotes a Sylow 3-subgroup of H_i

then Res : $H^*(H_i) \to H^*(P_i)$ is an isomorphism for $2 \leq i \leq 6$, so that $H^*(H_i)$ is free as an $H^*(\mathrm{SL}_2(3))$-module with basis 1. Therefore, $HH'(\mathbb{F}_3\,\mathrm{SL}_2(3))$ has generators defined as $C_i = \gamma_i(1)$ for $1 \leq i \leq 6$, $V = \gamma_1(v)$, and $U = \gamma_1(u)$. The only nontrivial relations are of Type III in degree zero, which are straightforward to obtain. $\square$

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